Fairness, distances and degrees

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Fairness, Distances and Degrees

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Abstract

We show the identity between sets of fair computations in recursive transition graphs, sets of cluster points of finite computations for Π_1^0 ultra-metrics refining the Baire metrics, and Π_3^0 subsets of ω^ω. The results are applied to recursive marked trees, fairness definitions, ω-regular languages, and Π_3^0 sets.

Equité, distances et degrés

Résumé

Nous identifions les ensembles d'exécutions équitables dans des graphes de transitions récursifs, les ensembles de points d'accumulation d'exécutions finies pour des ultra-distances Π_1^0 raffinant la distance de Baire, et les ensembles Π_3^0 de ω^ω. Ces résultats sont appliqués aux arbres marqués récursifs, aux définitions de l'équité, aux langages ω-réguliers, et aux ensembles Π_3^0.

I. Introduction

The dynamics of a machine or program is best represented as a transition graph, defined as a set of states equipped with a set of binary transitions between states, possibly labelled on an alphabet of transition symbols [Keller]. Runs of machines or programs are identified with countable paths in the graph, called computations. Most transition graphs encountered in practice are finite if they are models of machines, e.g. finite state automata [Hopcroft and Ullman], or recursive if they are models of programming calculi, e.g. the λ-calculus with β-reduction [Barendregt] or the Calculus of Communicating Systems [Milner]. Recursively enumerable transition graphs are a borderline case. A nice example is MEIJE, a synchronous process calculus with unguarded recursion [Austry-Boudol].

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In the case of deterministic programs, one is primarily interested in terminating computations, but infinite computations play a major role for deterministic machines, and thereby prompt the consideration of topologies. For instance, Landweber's theorem tells us that languages accepted by deterministic Büchi automata are G_δ (Π^0_2) in the topology induced by the standard ultra-metric distance on ω-words, because G_δ-sets coincide with Eilenberg limits of sets of finite words [Landweber] [Eilenberg]. With such automata, the set of successful computations generally differs from the set of infinite computations of which it is a subset. The set of infinite computations is the derived set of the set of finite computations in the natural metric topology on computations, derived from Baire metric, and thus it is Π^0_1. The subset of the successful computations is specified by a Π^0_2 predicate on infinite computations, and it is yet the derived set of the set of finite computations in another metric topology which refines the natural topology. The above situation is essentially reproduced for non deterministic machines, which accept ω-languages located higher in Borel's hierarchy (ω-regular languages are in the boolean closure of G_δ).

As regards programming calculi, a metric semantics for non deterministic recursive program schemes, based on the definition of a Baire like metric on computation trees, has been constructed in [Arnold-Nivat]. Similarly, a metric semantics for concurrent programs, based on the definition of a Baire like metric on streams, has been constructed in [de Bakker-Meyer]. That model is representative of unfairness in the following sense: the race between concurrent agents which compete for engaging themselves in an interleaved computation is totally free. The role of fairness is to place restrictions on that race, expressed by predicates on computations. The most popular forms of fairness are weak fairness and strong fairness [Francez]. In a weakly (resp. strongly) fair computation, an agent which is almost always (resp. infinitely often) enabled to act, acts infinitely often. If the set of agents is unbounded, the corresponding predicate on infinite computations is Π^0_2 for weak fairness and Π^0_3 for strong fairness. The set of fair computations is more complex than the set of infinite computations, which is Π^0_1 for recursive transition graphs as the derived set of the set of finite computations in the natural topology.

More essentially, for both weak or strong fairness, the set of fair computations is the derived set of the set of finite computations in a metric topology which refines the natural topology. This central property, shown first for weak fairness in the context of a special programming language [Degano-Montanari], was established later on for strong fairness in CCS [Costa]. In the light of the papers referred to, any definition of fairness aims at restricting the convergence of sequences of finite computations which approximate infinite computations: unfair computations, which disappear from the derived set of the set of finite computations, are then discarded. The parallel with sequential machines is almost perfect, and the question arises whether one can find a counterpart to Landweber's theorem.

In this paper, we will show that any Π^0_3 subset of ω^ω may be seen as the set of fair computations in a (fixed) recursive transition graph, for a (generic) concept of fairness based on a (varying) recursive relation which decides whether a given agent is enabled at a given step in a given computation. Furthermore, we will identify the Π^0_3 subsets of ω^ω with the family of the derived sets of ω^* for Π^0_1 metrics on ω^∞ refining the natural metric. As a result, the sets of fair computations in recursive transition graphs and the derived sets of ω^* for Π^0_1 metrics are the same. In addition, the correspondence between logical and metrical definitions of fairness is effective in both directions: definitions of Π^0_1 metrics (on ω^∞) are translated uniformly to definitions of fairness, and vice versa. That effective correspondence fulfills the research program which Degano and Montanari launched into. By relativizing the correspondence between Π^0_3 subsets of ω^ω and Π^0_1 metrics on ω^∞, we moreover obtain a metric characterization of the class $F_{\sigma\delta}$ ($\mathbf{\Pi}^0_3$) in the classical Borel hierarchy. This is the expected counterpart to Landweber's theorem.

The remaining sections of the paper are organized as follows. Notations and definitions are introduced in section II. The relationships between sets of fair computations and Π^0_3 sets of functions are studied in section III. The applications are considered in section IV.

II. Notations and definitions

In the sequel, ω is the set of natural numbers and ω^* is the set of finite sequences of natural numbers. We recall that $\omega^* = \bigcup_{k=0}^{\infty} \omega^k$ is mapped recursively onto ω by a one-one coding $\tau^*((x_1, \ldots, x_k)) = \langle x_1, \ldots, x_k, k-1\rangle + 1$, monotonous in each one of the x_i for sequences of fixed length $k>0$, where $\tau^*(\Phi) = 0$ [Rogers,p.71]. This coding gives rise to an isomorphism between the disjoint sums $\omega^\infty = \omega^* \cup \omega^\omega$ and $\Omega = \omega \cup \omega^\omega$ in the category of sets. For each $k \geq 0$ we can therefore define a function $[k]: \Omega \to \omega$ truncating objects at depth k according to the definitions:

$f[0] = 0$ *and* $f[k] = \tau^*((f(0), \ldots, f(k-1)))$ *for* $f \in \omega^\omega$ *and* $k>0$,

$x[0] = 0$ *and* $0[k] = 0$ *for* $x \in \omega$ *and* $k>0$,

$x[k] = \tau^*((x_1, \ldots, x_{\min(k,l)}))$ *for* $x = \tau^*((x_1, \ldots, x_l))$ *and* $k, l > 0$.

When we deal with arithmetical functions on Ω, and more generally with arithmetical relations on $\Omega^k \times \omega^l$, we define the arithmetical class of a relation R as the least upper bound (with respect to Kleene's hierarchy over Σ^0_n, Π^0_n and Δ^0_n classes) of the class of $R \cap ((X_1 \times \ldots \times X_i \times \ldots \times X_k) \times \omega^l)$ for variables X_i ranging over $\{\omega, \Omega\}$. We refer the reader to [Rogers] for a thorough presentation of the arithmetical hierarchy. Thus $\alpha[k]$ may be considered as a recursive function from $\Omega \times \omega$ to ω. By way of definition, $\alpha \wedge \beta$ is the least upper bound of the set $\{k \mid \alpha[k] = \beta[k]\}$.

The *natural metric* on Ω is the ultra-metric δ defined as $\delta(\alpha,\beta) = 0$ if $\alpha = \beta$, $1/1+(\alpha \wedge \beta)$ otherwise. Because the relation $\delta(\alpha, \beta) < 1/n$ is recursive in (α, β, n), we call δ a *recursive metric*. Similarly, a metric d on Ω is said to be Π^0_1 (resp. Σ^0_2) if the relation $d(\alpha,\beta) < 1/n$ is Π^0_1 (resp. Σ^0_2) in (α,β,n). We are mainly interested in the derived sets of ω and its recursive (or recursively enumerable) subsets for Π^0_1 metrics d on Ω refining δ ($d \geq \delta$). We recall from [Dugundji] that in a metric space (X, d), the *derived set* A' (or A'_d) of A is the set of the $x \in X$ such that any open ball with center x and radius $1/n$ contains at least one element $a \in A$ distinct from x.

We will show that any Π°_{3} subset of ω^{ω} coincides with ω'_{d} for some Π°_{1} metric d on Ω, representing a concept of effective fairness. Definitions of fairness make sense with respect to computations in transition graphs. A *transition graph* is a quadruple $T=(S,T,\sigma_{o},\sigma_{1})$ where S,T are sets of numbers (representing states and transitions) and σ_{o},σ_{1} are mappings from T to S (indicating the source and target of transitions). A transition graph is recursive (resp. recursively enumerable) if all its components are recursive (resp. Σ°_{1}). For instance, the transition graph $\omega=(\{o\},\omega,o,o)$ is recursive. A *pointed* transition graph T_S is a transition graph T with an initial state s.

A *computation* in a pointed transition graph T_{S_o} is a finite or denumerable sequence of transitions (t_i) satisfying $\sigma_o(t_i)=s_o$ if $i=o$ and $\sigma_o(t_i)=\sigma_1(t_{i-1})$ if $i>o$, for all i in the domain of the sequence. A finite computation $t_o \ldots t_{k-1}$ with length k is represented in Ω by the number $\tau^*((t_o,\ldots,t_{k-1}))$, while an infinite computation $(t_i)_{i\in\omega}$ is represented by the function $f(i)=t_i$. Let $\mathrm{Fin}(T_S)$ resp. $\mathrm{Inf}(T_S)$ denote the set of the finite resp. infinite computations from s in T, then, of course $\mathrm{Inf}(T_S)=(\mathrm{Fin}(T_S))'_{\delta}$ in the topology induced by the natural metric δ on Ω.

A concept of *effective fairness* in a transition graph T_S is totally determined by an *enabling predicate* defined as a recursive relation $E \subseteq \omega^2$. Intuitively, $E(f[k],i)$ means that agent i is enabled at the k^{th} step in computation f. An *E-fair* computation is then an infinite computation in which no agent is enabled infinitely often. In formulas, $[f \text{ is E-fair}] \Leftrightarrow [\forall i.\ \neg \overset{\omega}{\exists} k.\ E(f[k],i)]$, or, equivalently, $[f \text{ is E-fair}] \Leftrightarrow [\forall i.\ \overset{\omega}{\forall} k.\ \neg E(f[k],i)]$, where $\overset{\omega}{\exists}$ and $\overset{\omega}{\forall}$ are the dual infinite quantifiers. Let $\text{E-fair}(T_S)$ denote the set of E-fair computations in transition graph T_S.

For recursive (resp. recursively enumerable) transition graphs T_S, $\mathrm{Inf}(T_S)$ is Π°_{1} (resp. Π°_{2}), and $\text{E-fair}(T_S)$ is Π°_{3}. Similar observations pertain to the usual definitions of weak fairness (Π°_{2}), strong and extreme fairness (Π°_{3}), see for instance [Harel]. We will show that all those variant definitions of fairness may in fact be reduced to E-fairness without altering the transition graphs.

III. A connection between Π_3^0 sets of functions, Π_1^0 metrics on Ω, and E-fairness.

On support of the definitions and notations introduced in the last section, we can evolve a precise statement of the connection announced in the introduction.

Theorem 1 *For* $F \subseteq \omega^\omega$ *the following assertions 1, 2, 3 and 4 are equivalent:*

(1) F *is* Π_3^0 *(as a set of functions),*

(2) F *is the derived set of* ω *for some* Π_1^0 *ultra-metric on* Ω, *induced from a recursive distance on* ω *and refining the natural distance* δ,

(3) F *is the derived set of a* Σ_2^0 *subset of* ω *for some* Σ_2^0 *metric on* Ω,

(4) F *equals* E-fair(ω) *for some (recursive) enabling predicate* E.

The above characterization for Π_3^0 sets of functions is the main result of the paper. We state hereafter a variant characterization for Π_3^0 sets of infinite computations.

Theorem 2 *For any recursively enumerable (pointed) transition graph* T_S *the following assertions 5, 6 and 7 are equivalent:*

(5) F *is a* Π_3^0 *subset of* $\mathrm{Inf}(T_S)$,

(6) F *is the derived set of* $\mathrm{Fin}(T_S)$ *for some* Π_1^0 *ultra-metric refining* δ,

(7) F *equals* E-fair(T_S) *for some (recursive) enabling predicate* E.

In view of that adapted theorem, all the classical definitions of fairness may be reduced to the *universal* form of E-fairness. Fundamental for the proof of both theorems is the next lemma, which points out an analogous reduction for Π_3^0 definitions of sets of functions. This lemma extends a similar characterization for Π_3^0 sets of numbers, established by Kreisel, Shoenfield and Wang, see [Rogers].

Lemma 1 *(normal form for* Π_3^0 *sets of functions)*

Any Π_3^0 *subset* F *of* ω^ω *may be represented in the normal form*

$\{f \mid \forall i.\ \overset{\omega}{\forall} k.\ \neg E(f[k], i)\}$ *where* E *is a recursive binary relation on numbers, defined uniformly from a* Π_3^0*-index of* F.

<u>proof</u> Assume $F = \{f \mid \forall i.\ \exists j.\ \forall k.\ R(f,i,j,k)\}$ for some recursive relation R. For all i and j, the set $\bar{F}_{i,j}$ defined as $\{f \mid \exists k.\ \neg R(f,i,j,k)\}$ is clearly equivalent to $\{f \mid \exists k.\ \neg S(f[k],i,j)\}$ for some recursive relation S defined uniformly from R. For all i, the set F_i defined as $\{f \mid \exists j.\ \forall k.\ S(f[k],i,j)\}$ is in turn equal to $\{f \mid \exists j.\ \forall k \geq j.\ \neg E(f[k],i)\}$ for some recursive relation E defined uniformly from S. A decision procedure defining $E(x,i)$ is the following:

for k *in* $0:\omega$ *loop*

if $\forall h \leqslant k.\ S(x[h],i,j)$

then (*if* $x[k]=x$ *then return* $\neg E(x,i)$ *else* $j \leftarrow j$)

else (*if* $x[k]=x$ *then return* $E(x,i)$ *else* $j \leftarrow j+1$).

In fact, $f \in F_i$ if and only if the following process outputs eventually always 0, which was brought to our attention by A. Louveau:

for k *in* $0:\omega$ *loop*

if $\forall h \leqslant k.\ S(f[h],i,j)$

then (*output*(0) ; $j \leftarrow j$)

else (*output*(1) ; $j \leftarrow j+1$).

Altogether, $F = \{f \mid \forall i.\ \overset{\omega}{\forall} k.\ \neg E(f[k],i)\}$, and the result is obtained. □

Let us return to the main theorem. The above lemma may be read as $(1) \Rightarrow (4)$. If we can prove $(4) \Rightarrow (2)$, the theorem will follow from $(3) \Rightarrow (1)$ and $(2) \Rightarrow (3)$, which are immediate. The remaining implication $(4) \Rightarrow (2)$ is established below.

Assume $F = \{f \mid \forall i.\ \exists j.\ \forall k \geqslant j.\ \neg E(f[k],i)\}$. For $\alpha, \beta \in \Omega$ and $j \in \omega$, define:

(8) $\mathrm{done}(\alpha,j) = \max\{i \leqslant j \mid \forall i' \leqslant i.\ \forall k \geqslant j.\ \neg E(\alpha[k],i')\}$,

(where $\max \emptyset$ equals 0 by the usual convention)

(9) $d(\alpha,\beta) = 0$ if $\alpha = \beta$, and otherwise

$\max(1/1 + \mathrm{done}(\alpha, \alpha \wedge \beta)),\ 1/1 + \mathrm{done}(\beta, \alpha \wedge \beta)))$.

Intuitively, $\mathrm{done}(\alpha,j)$ is the number of the last agent whose all predecessors are *done* at stage j in computation α, for they are never enabled beyond that stage.

Now $\mathrm{done}(\alpha, j)$ is monotonously increasing in j, thus the following equivalence holds:

(10) $(f \in F) \Leftrightarrow \lim_{j \to \omega} \mathrm{done}(f, j) = \omega$.

On that basis, and assuming that d is a metric refining δ, we now prove:

(11) $F = \omega_d'$.

Assume $\alpha \in \omega_d'$ then by definition: $\forall j.\ \exists x_j \in \omega.\ 0 < d(\alpha, x_j) < 1/j$.

As $d \geqslant \delta$ and $(x_j)_j$ is a non stationary sequence of numbers, we know that $\alpha \in \omega^\omega$; then $\lim_{j \to \omega} d(\alpha, x_j) = 0 \Rightarrow \lim_{j \to \omega} \mathrm{done}(\alpha, \alpha \wedge x_j) = \omega \Rightarrow \lim_{j \to \omega} (\alpha \wedge x_j) = \omega$, and $\alpha \in F$ by (10).

Assume $f \in F$, then by (10): $\lim_{j \to \omega} \mathrm{done}(f, j) = \omega$.

$f \wedge f[j] = j \Rightarrow d(f, f[j]) = \max(1/1+\mathrm{done}(f, j), 1/1+\mathrm{done}(f[j], j))$;

$\left(j' \geqslant j \Rightarrow f[j] = (f[j])[j']\right) \Rightarrow \mathrm{done}(f[j], j) = \max\{i \leqslant j \mid \forall i' \leqslant i.\ \forall j' \geqslant j.\ \neg E((f[j])[j'], i')\}$
$= \max\{i \leqslant j \mid \forall i' \leqslant i. \neg E(f[j], i')\} \geqslant \max\{i \leqslant j \mid \forall i' \leqslant i. \forall j' \geqslant j. \neg E(f\ [j'], i')\} = \mathrm{done}(f, j)$;

thus $\lim d(f, f[j]) = 0$ entails $f \in \omega_d'$. $\Box$

In order to complete the proof of (4) $\Rightarrow$ (2) and thereby establish the main theorem, we still have to show that d is a Π_1^0 ultra-metric on Ω, induced from a recursive distance on ω and refining the natural metric δ. Relation $d(\alpha, \beta) \geqslant \delta(\alpha, \beta)$ is clear from the inequality $\mathrm{done}(\alpha, \alpha \wedge \beta) \leqslant \alpha \wedge \beta$. The following lemmas 2,3,4 provide the rest.

Lemma 2 *For* $\alpha, \beta, \gamma \in \Omega$: $d(\alpha, \beta) \leqslant \max\{d(\alpha, \gamma), d(\beta, \gamma)\}$.

proof The triangular inequality to be shown may be equivalently restated as

(12) $\min\{\mathrm{done}(\alpha, \alpha \wedge \beta), \mathrm{done}(\beta, \alpha \wedge \beta)\} \geqslant$

$\min\{\mathrm{done}(\alpha, \alpha \wedge \gamma), \mathrm{done}(\gamma, \alpha \wedge \gamma), \mathrm{done}(\beta, \beta \wedge \gamma), \mathrm{done}(\gamma, \gamma \wedge \beta)\}$.

We proceed by case analysis.

case 1. $\alpha \wedge \gamma = \beta \wedge \gamma \leqslant \alpha \wedge \beta$:

$\alpha \wedge \beta \geqslant \alpha \wedge \gamma \Rightarrow \mathrm{done}(\alpha, \alpha \wedge \beta) \geqslant \mathrm{done}(\alpha, \alpha \wedge \gamma)$,

$\alpha \wedge \beta \geqslant \beta \wedge \gamma \Rightarrow \mathrm{done}(\beta, \alpha \wedge \beta) \geqslant \mathrm{done}(\beta, \beta \wedge \gamma)$.

case 2. $\alpha \wedge \beta = \alpha \wedge \gamma \leqslant \beta \wedge \gamma$:

$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \mathrm{done}(\alpha, \alpha \wedge \beta) \geqslant \mathrm{done}(\alpha, \alpha \wedge \gamma)$,

and we claim that:

$\mathrm{done}(\beta, \alpha \wedge \beta) \geqslant \min\{\mathrm{done}(\beta, \beta \wedge \gamma), \mathrm{done}(\gamma, \alpha \wedge \gamma)\}$.

Suppose the contrary, and let us search for a contradiction. Define:

$I = 1 + \mathrm{done}(\beta, \alpha\wedge\beta)$,

then one may assert:

(13) $\exists k \geqslant \alpha\wedge\beta .\ E(\beta[k], I)$

because $I-1 = \mathrm{done}(\beta,\alpha\wedge\beta) < I$,

(14) $\forall k \geqslant \beta\wedge\gamma .\ \neg E(\beta[k], I)$

because $I \leqslant \mathrm{done}(\beta, \beta\wedge\gamma)$,

(15) $\forall k \geqslant \alpha\wedge\gamma .\ \neg E(\gamma[k], I)$

because $I \leqslant \mathrm{done}(\gamma, \alpha\wedge\gamma)$.

In view of relations $\alpha\wedge\gamma = \alpha\wedge\beta \leqslant \beta\wedge\gamma$ and $\gamma[\beta\wedge\gamma] = \beta[\beta\wedge\gamma]$, and by (15):

$\neg E(\beta[k], I)$ for any k satisfying $\alpha\wedge\beta \leqslant k < \beta\wedge\gamma$.

Hence a contradiction is reached between (13) and (14).

<u>case</u> 3. $\alpha\wedge\beta = \beta\wedge\gamma \leqslant \alpha\wedge\gamma$:

exchange α and β, and proceed as in case 2. $\square$

Lemma 3 d *is a* Π^0_1 *metric on* Ω, *recursive on* ω.

<u>proof</u> We will give equivalent Π^0_1 formulae for the three types of relations $d(f,g) < 1/n$, $d(f,x) < 1/n$, and $d(x,y) < 1/n$ (where $f,g \in \omega^\omega$ and $x,y,n \in \omega$).

By definition of d: $d(\alpha,\beta) < 1/n \Leftrightarrow$

(16) $\alpha = \beta \ \vee \ \big(\mathrm{done}(\alpha,\alpha\wedge\beta) \geqslant n \ \wedge \ \mathrm{done}(\beta,\alpha\wedge\beta) \geqslant n\big)$.

By definition of done: $\alpha = \beta \ \vee \ \mathrm{done}(\alpha,\alpha\wedge\beta) \geqslant n \Leftrightarrow$

(17) $\forall j. \big((\alpha[j] = \beta[j] \ \wedge \ \alpha[j+1] \neq \beta[j+1]) \ \rightarrow \ \forall i \leqslant n.\ \forall k \geqslant j.\ \neg E(\alpha[k], i) \big)$.

Thus $d(\alpha,\beta) < 1/n$ is a Π^0_1 relation.

As a matter of fact, the relations $d(f,g) < 1/n$, $d(f,x) < 1/n$, and $d(x,y) < 1/n$ are respectively Π^0_1, Π^0_1, and recursive (since $x = x[k] \Rightarrow x[k] = x[k+1]$). $\square$

Lemma 4 Let $\alpha,\beta \in \Omega$ be such that $\alpha \neq \beta$ then:

$d(\alpha,\beta) < 1/n \ \Leftrightarrow \ \exists p .\ \forall m \geqslant p .\ d(\alpha[m], \beta[m]) < 1/n$.

<u>proof</u> The above equivalence is trivially satisfied for $\alpha\wedge\beta < n$.

Suppose now $\alpha\wedge\beta = l \geqslant n$. By definition of d and done:

$$
\begin{aligned}
d(\alpha,\beta) \geqslant 1/n &\Leftrightarrow \left(\mathrm{done}(\alpha,\alpha\wedge\beta) < n \ \vee\ \mathrm{done}(\beta,\alpha\wedge\beta) < n\right)\\
&\Leftrightarrow \exists i \leqslant n.\ \exists k \geqslant l.\ \left(E(\alpha[k],i) \vee E(\beta[k],i) \right)\\
&\Leftrightarrow \exists p.\ \forall m \geqslant p.\ \exists i \leqslant n.\ \exists k \geqslant l.\\
&\qquad \left(E(\alpha[m],i) \vee E(\beta[m],i)\right) \vee \left(k \leqslant m \wedge (E(\alpha[k],i) \vee E(\beta[k],i)) \right)\\
&\Leftrightarrow \exists p.\ \forall m \geqslant p.\ \exists i \leqslant n.\ \exists k \geqslant l.\ \left(E(\alpha[m][k],i) \vee E(\beta[m][k],i) \right)\\
&\Leftrightarrow \exists p.\ \forall m \geqslant p.\ \exists i \leqslant n.\ \exists k \geqslant \min(l,m).\ \left(E(\alpha[m][k],i) \vee E(\beta[m][k],i) \right)\\
&\Leftrightarrow \exists p.\ \forall m \geqslant p.\ d(\alpha[m],\beta[m]) \geqslant 1/n \quad \square
\end{aligned}
$$

The proof for theorem 1 is complete. We now sketch a proof for theorem 2. Let T_S be a recursively enumerable (pointed) transition graph. Since $\mathrm{Inf}(T_S)$ is the derived set of $\mathrm{Fin}(T_S)$ for the natural metric δ, (5) $\Rightarrow$ (6) $\Rightarrow$ (7) by straightforward adaptation of (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Finally, (7) $\Rightarrow$ (5) is immediate.

As a concluding remark, let us underline the uniform construction of d from **F**, and conversely. Applications are discussed in the final section.

IV. Applications

In the above section, we have identified the Π^0_3 subsets of ω^ω with the derived sets of ω ($\equiv \omega^*$) in metric topologies induced by Π^0_1 distances on Ω ($\equiv \omega^* \cup \omega^\omega$). We will now examine applications and extensions. The main fields of application gone through are recursive marked trees, fairness expressions, and ω-regular languages. The climax is a metric characterization of the family $F_{\sigma\delta}$ ($\mathbf{\Pi}^0_3$) in the classical Borel hierarchy, established in two different ways.

Recursive marked trees

In [Harel] was defined a language for stating properties of infinite paths in recursive marked trees. An ω-tree T is a subset of ω^* closed under left factors, represented by a corresponding set of numbers $t = \{\tau^*((x_1,\ldots,x_k)) \mid x_1\ldots x_k \in T\}$.

The set of infinite paths in T, represented by the set of functions $\{f \in \omega^{\omega} \mid \forall k.\, f[k] \in t\}$, is Π^0_1 if t is a recursive tree. A *recursive marked tree* $(t, \mathfrak{m})$ is a recursive tree t whose set of leaves is recursive and whose nodes are labelled by (possibly infinite) sets of numbers, fixed by a recursive predicate $\mathfrak{m} \subseteq t \times \omega$.

Infinite paths in a recursive tree may be seen as infinite computations in a recursive transition graph with a recursive set of sink states. Marks may be thought of as identifiers for computing agents and in the sequel, we shall interpret $\mathfrak{m}(f[k], a)$ as the affirmation that agent a is *disabled* at the k^{th} step in computation f. The complement $\overline{\mathfrak{m}}$ of a marking predicate $\mathfrak{m}$ is thus an enabling predicate.

Harel's language L is the union of an alternated hierarchy of languages L_i, L'_i constructed as follows. The basic language L_o provides four species of atomic formulas $\exists a$, $\forall a$, $\exists^{\infty} a$, $\forall^{\infty} a$ where a ranges over the set of marks $(a \in \omega)$. For $i \geqslant o$, L'_i is the closure of L_i under finite conjunction and disjunction and under recursive ω-conjunction, and L_{i+1} is the closure of L'_i under recursive ω-disjunction. Formulas in L are interpreted over infinite pathes in recursive marked trees. An infinite path f in $(t, \mathfrak{m})$ satisfies the formula $\exists a$, respectively $\exists^{\infty} a$, if and only if $\mathfrak{m}(f[k], a)$ for some k, respectively $\mathfrak{m}(f[k], a)$ for infinitely many k. Formulas $\forall a$, $\forall^{\infty} a$ are the duals of $\exists a$, $\exists^{\infty} a$ and the logical connectives have the standard interpretation.

In a recursive marked tree $(t, \mathfrak{m})$, the atomic formulas $\forall a$, $\exists a$, $\forall^{\infty} a$, $\exists^{\infty} a$ are interpreted respectively by Π^0_1, Σ^0_1, Σ^0_2, Π^0_2 sets of infinite paths, and each formula $\varphi \in L'_o$ is therefore interpreted by a Π^0_3 set of infinite paths. Conversely, any Π^0_3 subset of ω^{ω} may be represented as $\{f \mid \forall i.\, \overset{\omega}{\forall} k.\, \mathfrak{m}(f[k], i)\}$ for some recursive relation $\mathfrak{m}$, and thus coincides with the interpretation of the formula $\bigwedge_i \forall^{\infty} i$ in the recursive marked tree $(\omega, \mathfrak{m})$. Altogether, for $n > o$, a subset of ω^{ω} is Σ^0_{2n+2} resp. Π^0_{2n+3} if and only if it coincides with the interpretation of some formula $\varphi \in L_n$ resp. $\varphi \in L'_n$ in some recursive marked tree $(t, \mathfrak{m})$. Furthermore, the tree and the marking predicate are defined uniformly from the Σ^0_i or Π^0_i index of the set.

Logical expressions of fairness

In view of the above, there are at least three equivalent ways of defining fairness in recursively enumerable transition graphs:

i) state an arbitrary Π^0_3 predicate acting as a filter on infinite computations,

ii) state a recursive distance d on finite computations, refining the natural distance δ, and define fair computations as natural limits of d-Cauchy sequences of finite computations,

iii) state a recursive marking / enabling predicate $\mathfrak{m}$ for finite computations, and rest on the general normal form $F = \{f \mid \forall i. \overset{\omega}{\forall} k.\ \mathfrak{m}(f[k], i)\}$ for Π^0_3 sets.

Furthermore, there is a uniform translation between any two types of definitions of fairness, and each type of definitions may be equipped with effective operators realizing the conjunction, disjunction, or ω-conjunction of fairness conditions. For instance, the union of the derived sets of ω for two different Π^0_1 distances d_1 and d_2 on Ω is again the derived set of ω for some Π^0_1 distance $d_1 \vee d_2$ on Ω. That specific property of Π^0_1 distances is not trivial, and we have not heard about similar cases in topology.

ω-regular languages

A well known theorem due to Mac Naughton states that the family of ω-regular languages over A is the boolean closure of the family of deterministic ω-regular languages over A [Mac Naughton]. Since any deterministic ω-regular language is Π^0_2, any ω-regular language is Π^0_3. Thus, any ω-regular language over A is the derived set of A^* for some Π^0_1 metric distance on $A^\infty = A^* \cup A^\omega$. This fact is not surprising, since recognition criteria used in Büchi / Müller automata are essentially fairness conditions.

A metric characterization of $F_{\sigma\delta}$ (Π^0_3)

There exists a tight relationship between the classical Borel hierarchy of sets of functions and the effective Kleene hierarchy, see for instance [Rogers, p.356] and [Moschovakis, p.160]. Namely, if Γ is a (lightface) pointclass in the effective hierarchy and $\boldsymbol{\Gamma}$ is the corresponding (boldface) pointclass in the classical hierarchy, then:

$$\boldsymbol{\Gamma} = \bigcup_{\varepsilon \in \omega^\omega} \Gamma[\varepsilon],$$

where a set of functions is $\Gamma[\varepsilon]$ iff it is Γ *relative* to the oracle ε. More precisely, $\Sigma^0_n[\varepsilon]$-forms and $\Pi^0_n[\varepsilon]$-forms are defined in the same way as Σ^0_n-forms and Π^0_n-forms, except that recursive relations are replaced by relations *recursive in* ε.

Thus, by relativizing theorem 1 to an arbitrary oracle ε, we get that a set of functions is $\boldsymbol{\Pi}^0_3$ if and only if it is the derived set of ω for some $\boldsymbol{\Pi}^0_1$ ultra-metric on Ω, refining the natural metric δ. A more accurate characterization of $\boldsymbol{\Pi}^0_3$ sets may be given in terms of *inductive* distances on Ω, defined as follows: a distance d on Ω is inductive if it takes values in $\{0\} \cup \{1/n \mid n\in\omega\}$, is weakly continuous in the sense that $d(\alpha,\beta) = \lim_k d(\alpha[k],\beta[k])$, and has only δ-closed d-balls.

Theorem 3 *For* $F \subseteq \omega^\omega$ *the following assertions 1,2 and 3 are equivalent:*

(1) F *is a* $\boldsymbol{\Pi}^0_3$ *set (or* F *is* $F_{\sigma\delta}$*),*

(2) F *is the derived set of* ω *for some inductive distance on* Ω,

(3) F *is the derived set of* ω *for some inductive ultra-metric refining* δ.

<u>proof</u> First of all, let us recall that the class of δ-closed sets is just $\boldsymbol{\Pi}^0_1$, see [Rogers, p.342] or [Moschovakis, p.20]. The proof is then straightforward.

<u>Suppose</u> (1), then F is $\Pi^0_3[\varepsilon]$ for some $\varepsilon \in \omega^\omega$. A remake of the proof for theorem 1 relative to ε shows that F is the derived set of ω for some $\Pi^0_1[\varepsilon]$ distance d on Ω, taking values $(1/n)$ and furthermore weakly continuous. Now, for each α, each d-ball $\{\beta \mid d(\alpha,\beta) < 1/n\}$ is $\Pi^0_1[\gamma]$ for some γ determined from ε and α, hence it is δ-closed. Thus $(1) \Rightarrow (2)$.

Suppose (2). There suffices to show that $d(\alpha,\beta) < 1/n$ is Σ^0_2 in (α,β,n), which entails (1). By definition of an inductive distance, $d(\alpha,\beta) < 1/n \Leftrightarrow \exists i.\ \forall j \geq i.\ d(\alpha[j], \beta[j]) < 1/n$. For $x \in \omega$, each d-ball $\{\beta \mid d(x,\beta) < 1/n\}$ is closed and thus $\Pi^0_1[\varepsilon_{x,n}]$ for some $\varepsilon_{x,n}$ in ω^ω. Hence $d(x,y) < 1/n$ is $\Pi^0_1[\varepsilon]$ for some global oracle ε gathering the $\varepsilon_{x,n}$, and $d(\alpha,\beta) < 1/n$ is $\Sigma^0_2[\varepsilon]$ and thus Σ^0_2 in (α,β,n). Thus (2) $\Rightarrow$(1).

Finally, (2) $\Leftrightarrow$(3) by the relativized version of theorem 1. $\square$

An elegant proof for (1) $\Rightarrow$(3) by purely topological treatment was devised by A. Arnold, yielding an independent proof of the theorem, as (3) $\Rightarrow$(1) is immediate. We sketch here the construction of the distance. Let F be a Π^0_3-subset of ω^ω, thus $F = \{f \mid \forall i.\ \exists j.\ \forall k.\ R(f,i,j,k)\}$ for some relation R recursive in ε for some oracle ε. Define $F_{i,j} = \{f \mid \exists j' \leq j.\ \forall k.\ R(f,i,j',k)\}$ and $F'_{i,j} = F_{i,j} \cup \{x \in \omega \mid \exists f.\ x \sqsubset f \in F_{i,j}\}$ where $x \sqsubset f$ if x is a prefix of f, i.e. $x = 0$ or $x = \tau^*((f(0),\dots,f(k)))$ for some $k \geq 0$. For each $i \in \omega$, the $F'_{i,j}$ form an increasing sequence of δ-closed subsets of Ω. Now for any $\alpha \in \Omega$, define $g_\alpha(i) = \inf\{j \mid \alpha \in F'_{i,j}\}$, where $\inf(\emptyset) = \infty$ and $\underline{\infty \notin \omega}$, then $g_\alpha(i) = \lim_k (g_{\alpha[k]}(i))$, and $(\alpha \in F)$ if and only if $(\alpha \in \omega^\omega$ and $g_\alpha \in \omega^\omega)$. The distance $d(\alpha,\beta)$ is finally defined as $1/1+\Delta(\alpha,\beta)$ letting $\Delta(\alpha,\alpha) = \infty$ and for $\alpha \neq \beta$: $\Delta(\alpha,\beta) = \max\{n \leq \alpha \wedge \beta \mid g_\alpha[n] = g_\beta[n] = g_{\alpha\cap\beta}[n] \in \omega^n\}$ where $\alpha \cap \beta = \alpha[\alpha \wedge \beta] = \beta[\alpha \wedge \beta]$. Then d is indeed an inductive ultra-metric and F is the derived set of ω for that metric.

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